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DISTANCE ANTIMAGIC LABELING FOR PANCYCLIC GRAPHS

N. P. Shrimali, Y. M. Parmar and M. A. Patel*

Department of Mathematics,
Gujarat University, Ahmedabad, Gujarat, INDIA

E-mail : narenp05@gmail.com, ymp.maths@gmail.com

*Department of Mathematics,
Government Engineering College, Gandhinagar, Gujarat, INDIA

E-mail : dr.mahendraapatel@gmail.com

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Abstract: A distance antimagic labeling of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a bijection from vertex set $V(G)$ to $\{1, 2, ..., |V(G)|\}$ such that $\sum_{p\in N(q)} f(p) = w(q)$ for all $q \in V(G)$, where $N(q)$ is the set of all vertices of $V(G)$ which are adjacent to q and $w(p) \neq w(q)$ for every pair of vertices $p, q \in V(G)$. A graph which admits a distance antimagic labeling is called a distance antimagic graph. In this paper, we addresses distance antimagic labeling of some specific pancyclic graphs.

Keywords and Phrases: Distance antimagic labeling, pancyclic graph.

2020 Mathematics Subject Classification: 05C78.

1. Introduction

Here, we consider all graphs G with vertex set $V(G)$ and edge set $E(G)$ are finite and simple. $|V(G)|$ and $|E(G)|$ denote the number of vertices and number of edges respectively. Gross and Yellen [5] is adopted for the comprehension of the graph theoretical terminologies and for number theoretical results, we follow Burton [3]. For acquiring the latest update, we follow a dynamic survey on graph labeling by Gallian [4].

Definition 1.1. *A distance antimagic labeling of a graph G is a bijection* $f : V(G) \to \{1, 2, ..., |V(G)|\}$ *such that* $\sum_{p \in N(q)} f(p) = w(q)$ *for all* $q \in V(G)$*, where* $N(q)$ *is the set of all vertices of* $V(G)$ *which are adjacent to* q *and* $w(p) \neq w(q)$ *for every pair of vertices* $p, q \in V(G)$*. A graph which admits a distance antimagic labeling is called a distance antimagic graph.*

A distance antimagic labeling is introduced by N. Kamatchi and S. Arumugam [1] in 2013. The following Lemma has proved by Simanjuntak and Wijaya [9] in 2013.

Lemma 1.2. *If a graph contains two vertices with the same neighborhood then it is not distance antimagic.*

A few results are listed below which have been proved in [1].

- The cycle C_n is a distance antimagic for $n \neq 4$.
- The wheel W_n is a distance antimagic for $n \neq 4$.
- The path P_n is a distance antimagic.
- The graph $rK_2 + K_1$ is a distance antimagic.
- For any graph G of order n, the corona $G \odot K_1$ is a distance antimagic.

Shrimali and Parmar [10] have proved some products between cycle with four vertices (C_4) and friendship graph (C_3^t) like $C_3^t \,\square\, C_4$, $C_3^t \boxtimes C_4$, $C_4 \odot C_3^t$ are distance antimagic graphs. In the present paper we deal with some specific pancyclic graphs.

Definition 1.3. *A graph G with vertex set* $V(G)$ *and edge set* $E(G)$ *is called pancyclic graph if it contains the cycle of all orders up to* $|V(G)|$*.*

The concept of a pancyclic graph was first investigated in the context of tournaments by Harary and Moser [6]. Then pancyclicity was named and extended to undirected graphs by Bondy [2]. Pancyclic graphs are generalization of Hamiltonian graphs, graphs which have a cycle of the maximum possible length.

In the next section, we work on some specific pancyclic graphs.

2. Main Results

Jia-Bao Liu, H.U. Afzal and E. Bonyah [8] have defined a few specific pancyclic graphs. Aditionaly, M. Javaid, H. U. Afzal and E. Bonyah [7] have defined a few more pancyclic graphs. In this section, we assign distance magic labeling on specific pancyclic graphs.

Definition 2.1. *A pancyclic graph $\mathcal{H}_1$ for $n \geq 3$, having the construction as follows :*

$$V(\mathcal{H}_1) = \{v_i, u_i/1 \leq i \leq n\} \cup \{x_1, x_2\}$$

and

$$\begin{aligned} E(\mathcal{H}_1) \ &= \{v_iv_{i+1}, u_iu_{i+1}, v_iu_{i+1}/1 \leq i \leq n-1\} \\ &\cup \{v_iu_i/1 \leq i \leq n\} \cup \{v_1x_1, u_1x_1, v_nx_2, u_nx_2\}; \end{aligned}$$

with $|V(\mathcal{H}_1)| = 2n+2$ and $|E(\mathcal{H}_1)| = 4n+1$.

Theorem 2.2. *A pancyclic graph $\mathcal{H}_1$ is a distance antimagic graph.*

Proof. We define a vertex labeling $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ as follows :

$$\begin{aligned} f(v_i) \ &= 2i+1; \ i = 1, 2, \ldots, n \\ f(u_i) \ &= 2i; \qquad i = 1, 2, \ldots, n \\ f(x_1) \ &= 1, \\ f(x_2) \ &= 2n+2. \end{aligned}$$

Under the above labeling, we get weights for each vertex as follows :

$$\begin{aligned} w(v_i) \ &= \begin{cases} 8i+4; & i = 1, 2, \ldots, n-1 \\ 6i+1; & i = n \end{cases} \\ w(u_i) \ &= 8i; \qquad\qquad i = 1, 2, \ldots, n \\ w(x_i) \ &= \begin{cases} 4i+1; & i = 1 \\ 4n+1; & i = 2. \end{cases} \end{aligned}$$

Since, weights of each vertex are distinct, $\mathcal{H}_1$ is a distance antimagic graph.

Illustration 2.3. *Distance antimagic labeling for $\mathcal{H}_1$ with $n = 6$.*

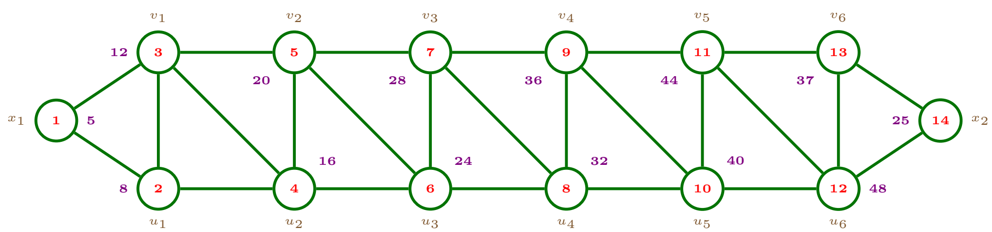

Figure 1: $\mathcal{H}_1$ with $n = 6$

Definition 2.4. *A pancyclic graph $\mathcal{H}_2$ for $n \geq 3$, having the vertex set and edge set as follows :*

$$V(\mathcal{H}_2) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{x_1, x_2\}$$

and

$$\begin{aligned} E(\mathcal{H}_2) &= \{v_i v_{i+1}, u_i u_{i+1}, v_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i u_i / i = 1, n\} \\ &\cup \{v_1 x_1, u_1 x_1, v_n x_2, u_n x_2, x_1 x_2\}; \end{aligned}$$

with $|V(\mathcal{H}_2)| = 2n + 2$ and $|E(\mathcal{H}_2)| = 3n + 4$.

Theorem 2.5. *A pancyclic graph $\mathcal{H}_2$ is a distance antimagic graph.*
Proof. We define a vertex labeling $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ by following two cases:
Case - 1. $n \not\equiv 2(mod3)$

$$\begin{aligned} f(v_i) &= 2i + 1; \ i = 1, 2, \ldots, n \\ f(u_i) &= 2i; \qquad i = 1, 2, \ldots, n \\ f(x_1) &= 1, \\ f(x_2) &= 2n + 2. \end{aligned}$$

Under the above labeling, we get weights for each vertex as follows :

$$\begin{aligned} w(v_i) &= \begin{cases} 12; & i = 1 \\ 6i + 4; & i = 2, 3, \ldots, n-1 \\ 6i + 1; & i = n. \end{cases} \\ w(u_i) &= \begin{cases} 8; & i = 1 \\ 6i - 1; & i = 2, 3, \ldots, n-1 \\ 8i; & i = n. \end{cases} \\ w(x_1) &= 2n + 7, \\ w(x_2) &= 4n + 2. \end{aligned}$$

One can easily verify that, all weights are distinct.

Case - 2. $n \equiv 2(mod3)$

$$
\begin{aligned}
f(v_i) &= 2i+1; \qquad i = 1,2,\ldots,n \\
f(u_i) &= \begin{cases} 2i; & i = 1,2,\ldots,n-2 \\ 2i+2; & i = n-1 \\ 2i-2; & i = n \end{cases} \\
f(x_1) &= 1, \\
f(x_2) &= 2n+2.
\end{aligned}
$$

By applying the above labeling, we get weights for each vertex as follows :

$$
\begin{aligned}
w(v_i) &= \begin{cases} 12; & i = 1 \\ 6i+4; & i = 2,3,\ldots,n-3 \\ 6i+6; & i = n-2 \\ 6i+2; & i = n-1 \\ 6i-1; & i = n. \end{cases} \\
w(u_i) &= \begin{cases} 8; & i = 1 \\ 6i-1; & i = 2,3,\ldots,n-3 \\ 6i+1; & i = n-2 \\ 6i-3; & i = n-1 \\ 8i+2; & i = n. \end{cases} \\
w(x_1) &= 2n+7, \\
w(x_2) &= 4n.
\end{aligned}
$$

Here also we get all distinct weights for case-2.
Hence, by case-1 and case-2 we can say that the graph $\mathcal{H}_2$ is a distance antimagic graph.

Illustration 2.6. *Distance antimagic labeling for $\mathcal{H}_2$ with $n = 6$.*

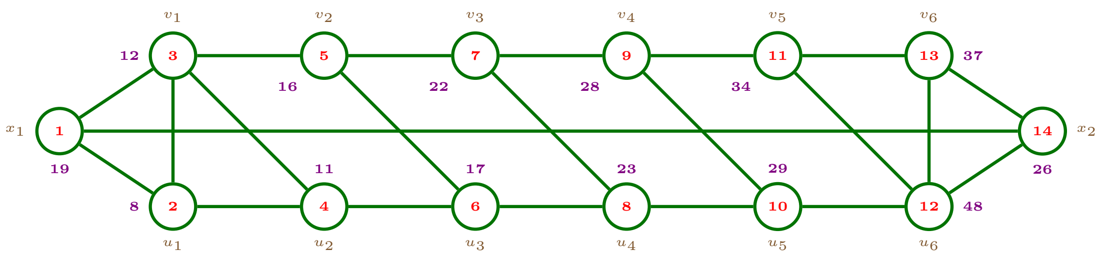

Figure 2: $\mathcal{H}_2$ with $n = 6$

Illustration 2.7. *Distance antimagic labeling for $\mathcal{H}_2$ with $n = 5$.*

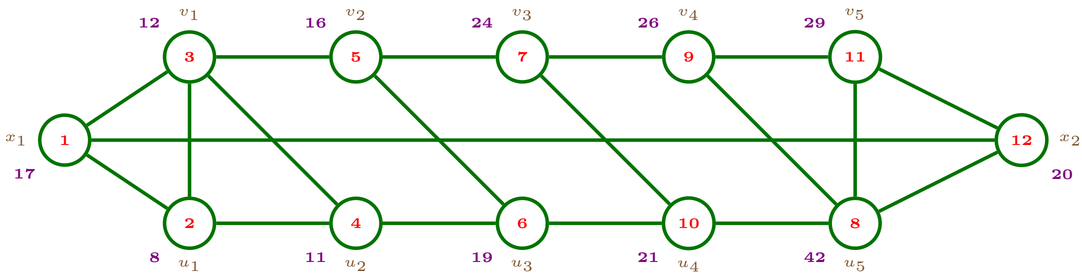

Figure 3: $\mathcal{H}_2$ with $n = 5$

Definition 2.8. *A pancyclic graph $\mathcal{H}_3$ for $n \geq 3$ having the vertex set $V(\mathcal{H}_3)$ and edge set $E(\mathcal{H}_3)$ as follows :*

$$V(\mathcal{H}_3) = \{v_i, u_i / 1 \leq i \leq n\} \cup \{x_i / 1 \leq i \leq 2n\}$$

and

$$\begin{aligned} E(\mathcal{H}_3) &= \{v_i v_{i+1}, u_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i x_{2i}, u_i x_{2i} / 1 \leq i \leq n\} \\ &\cup \{v_i x_{2i-1}, u_i x_{2i-1} / 1 \leq i \leq n\} \cup \{x_i x_{i+1} / 1 \leq i \leq 2n-1\}; \end{aligned}$$

with $|V(\mathcal{H}_3)| = 4n$ and $|E(\mathcal{H}_3)| = 8n - 3$.

Theorem 2.9. *A pancyclic graph $\mathcal{H}_3$ is a distance antimagic graph.*
Proof. We define a vertex labeling $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ as follows :

$$\begin{aligned}
f(v_i) &= 4i - 2; \qquad i = 1, 2, \ldots, n-1 \\
f(v_n) &= \begin{cases} 4n; & n \neq 3 \\ 4n - 2; & n = 3 \end{cases} \\
f(u_i) &= 4i; \qquad i = 1, 2, \ldots, n-1 \\
f(u_n) &= \begin{cases} 4n - 1; & n \neq 3 \\ 4n - 3; & n = 3 \end{cases} \\
f(x_{2i-1}) &= 4i - 1; \qquad i = 1, 2, \ldots, n-1 \\
f(x_{2n-1}) &= \begin{cases} 4n - 2; & n \neq 3 \\ 4n - 1; & n = 3 \end{cases} \\
f(x_{2i}) &= 4i - 3; \qquad i = 1, 2, \ldots, n-1 \\
f(x_{2n}) &= \begin{cases} 4n - 3; & n \neq 3 \\ 4n; & n = 3. \end{cases}
\end{aligned}$$

Under the above labeling, we get weights for each vertex as follows :

$$
\begin{aligned}
w(v_i) &= \begin{cases} 10i; & i = 1 \\ 16i - 8; & i = 2, 3, \ldots, n-2 \end{cases} \\
w(v_{n-1}) &= \begin{cases} 16n - 22; & n \neq 3 \\ 16n - 24; & n = 3 \end{cases} \\
w(v_n) &= \begin{cases} 12n - 11; & n \neq 3 \\ 12n - 7; & n = 3 \end{cases} \\
w(u_i) &= 16i - 4; \qquad i = 1, 2, \ldots, n-2 \\
w(u_{n-1}) &= \begin{cases} 16n - 21; & n \neq 3 \\ 16n - 23; & n = 3 \end{cases} \\
w(u_n) &= \begin{cases} 12n - 9; & n \neq 3 \\ 12n - 5; & n = 3 \end{cases} \\
w(x_{2i-1}) &= \begin{cases} 7; & i = 1 \\ 16i - 12; & i = 2, 3, \ldots, n-1 \end{cases} \\
w(x_{2n-1}) &= \begin{cases} 16n - 11; & n \neq 3 \\ 16n - 12; & n = 3 \end{cases} \\
w(x_{2i}) &= 16i; \qquad i = 1, 2, \ldots, n-2 \\
w(x_{2(n-1)}) &= \begin{cases} 16n - 17; & n \neq 3 \\ 16n - 16; & n = 3. \end{cases} \\
w(x_{2n}) &= \begin{cases} 12n - 3; & n \neq 3 \\ 12n - 6; & n = 3. \end{cases}
\end{aligned}
$$

Since, weights of each vertex are distinct, $\mathcal{H}_3$ is a distance antimagic graph.

Illustration 2.10. *Distance antimagic labeling for $\mathcal{H}_3$ with $n = 3$.*

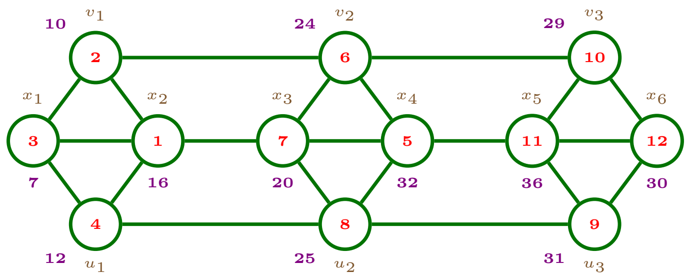

Figure 4: $\mathcal{H}_3$ with $n = 3$

Illustration 2.11. *Distance antimagic labeling for $\mathcal{H}_3$ with $n = 5$.*

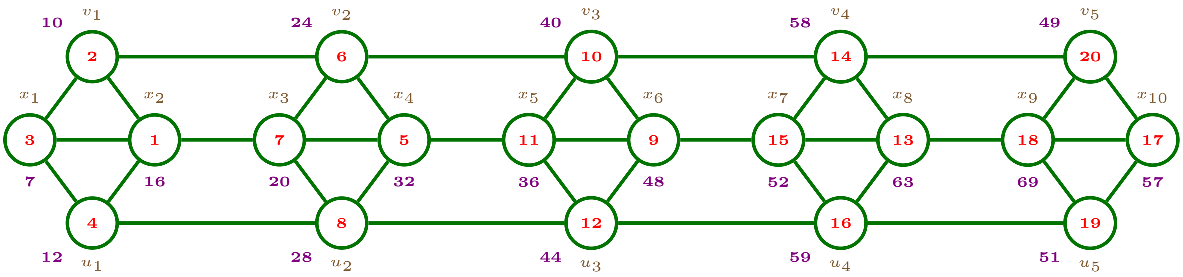

Figure 5: $\mathcal{H}_3$ with $n = 5$

The frucht graph was first described by Robert Frucht in 1939. This graph is a pancyclic graph.

Definition 2.12. *The frucht graph is a 3-regular graph with 12 vertices, 18 edges and no symmetries.*

Theorem 2.13. *The Frucht graph is a distance antimagic graph.*
Proof. We define a vertex labeling $f : V(G) \to \{1, 2, \ldots, 12\}$ as follows :

$$f(v_i) = i; \ \forall i$$

Under the above labeling, we get all vertex weights are distinct, as shown in the Figure 6.

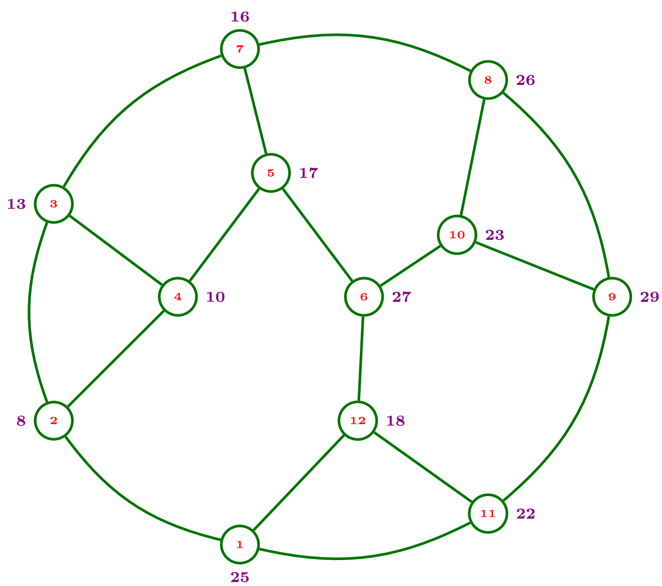

Figure 6: Frucht graph

3. Conclusion

Here, we have proved some specific pancyclic graphs and Frucht graph which is also a pancyclic graph are distance antimagic graphs. To define some new pancyclic graphs is an open problem and also to explore some new distance antimagic graph is an open problem.

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